On the superintegrability of the rational Ruijsenaars-Schneider model

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Abstract

The rational and hyperbolic Ruijsenaars-Schneider models and their non-relativistic limits are maximally superintegrable since they admit action variables with globally well-defined canonical conjugates. In the case of the rational Ruijsenaars-Schneider model we present an alternative proof of the superintegrability by explicitly exhibiting extra conserved quantities relying on a generalization of the construction of Wojciechowski for the rational Calogero model.

1 Introduction

Let us consider a Hamiltonian system (M, ω, h), where (M, ω) is a symplectic manifold of dimension $2n$ and h is the Hamiltonian. The system is *Liouville integrable* if there exist n independent functions $h_i \in C^\infty(M)$ $(i = 1, \dots, n)$ that are in involution with respect to the Poisson bracket and the Hamiltonian h is equal to one of the h_i. A Liouville integrable system is called *maximally superintegrable* if it admits $(n-1)$ additional constants of motion, say $f_j \in C^\infty(M)$, that are time-independent, globally smooth, and the $(2n-1)$ functions

$$h_1, \dots, h_n, f_1, \dots, f_{n-1} \tag{1}$$

are independent, i.e., their differentials are linearly independent on a dense submanifold of M. As a general reference on superintegrability, we refer to [1]. Maximal superintegrability and the compactness of the joint level surfaces

$$h_i = h_i^0 \quad (\forall i = 1, \dots, n) \quad \text{and} \quad f_j = f_j^0 \quad (\forall j = 1, \dots, n-1) \tag{2}$$

for generic constants h_i^0 and f_j^0 implies the periodicity of the generic flows of the system (M, ω, h). Maximally superintegrable systems with periodic flows are very rare, the classical examples being the isotropic harmonic oscillator and the negative energy sector of the Kepler problem. Other examples are provided by magnetic analogues and higher dimensional generalizations of the Kepler problem [2]. See also [3, 4] for interesting classification results in low dimensions.

There exists a large class of Liouville integrable systems that are maximally superintegrable in a rather obvious manner. These are the systems that admit action variables with globally well-defined canonical conjugates. Typical examples are scattering systems having configuration space trajectories $q(t) = (q_1(t), \dots, q_n(t))$ with large time asymptotes of the form

$$q_i(t) \sim q_i^+ + t v_i^+ \quad (\forall i = 1, \dots, n) \qquad \text{as} \quad t \to \infty \tag{3}$$

in such a way that the action variables $p_i^+ = m_i v_i^+$ and their canonical conjugates belong to $C^\infty(M)$ and together they parametrize the phase space M. This is the expected behaviour in many-body models of particles interacting via repulsive pair potentials. For example, the scattering characteristics of the rational and hyperbolic Calogero-Sutherland models [5, 6, 7] as well as of their Ruijsenaars-Schneider (RS) deformations [8] have been analyzed in [9], and the results imply that these many-body models are maximally superintegrable.

In our opinion, it is interesting to find explicit expressions for the constants of motion even for those systems whose maximal superintegrability is already known from abstract arguments. This is especially true if simple expressions can be obtained. A nice example is the direct proof of the maximal superintegrability of the rational Calogero model presented by Wojciechowski [10]. His arguments were later generalized to the rational Calogero models based on arbitrary finite Coxeter groups [11]. An attempt to study the hyperbolic Sutherland model in a similar manner was made in [12], but in this case the constants of motion have a more complicated structure and to our knowledge fully explicit globally smooth expressions of them are still not available.

The main goal of this letter is to explicitly exhibit constants of motion that show the maximal superintegrability of the rational Ruijsenaars-Schneider model. We find that the method of

Wojciechowski can be successfully applied in this case. Before explaining this in Section 3, in the next section we present a general argument implying the maximal superintegrability of the Liouville integrable systems that admit action variables with globally well-defined canonical conjugates. The precise definition of the foregoing condition is given below.

2 A class of maximally superintegrable systems

Consider the Liouville integrable systems (M, ω, h_i) associated with the Poisson commuting, independent Hamiltonians $h_1, \ldots, h_n$. Let us assume that globally well-defined action variables with globally well-defined canonical conjugates exist. By definition, this means that we have a phase space $(\mathcal{M}, \Omega)$ of the form

$$\mathcal{M} := \mathbb{R}^n \times \mathcal{D}_n = \{(Q, P) \,|\, Q \in \mathbb{R}^n,\ P \in \mathcal{D}_n\} \tag{4}$$

with a connected open domain $\mathcal{D}_n \subseteq \mathbb{R}^n$ and canonical symplectic form

$$\Omega = \sum_{i=1}^{n} dP_i \wedge dQ_i, \tag{5}$$

which is symplectomorphic to (M, ω) and permits identification of the Hamiltonians $\{h_i\}$ as functions of the action variables $\{P_j\}$. More precisely, we assume the existence of a symplectomorphism

$$A : M \to \mathcal{M} \tag{6}$$

such that the functions $\mathcal{H}_i := h_i \circ A^{-1}$ do not depend on the variables $\{Q_j\}$ and

$$X_{i,j} := \frac{\partial \mathcal{H}_i}{\partial P_j} \tag{7}$$

yields an invertible matrix $X(P)$ at every $P \in \mathcal{D}_n$. The map A is referred to as a global action-angle map of maximally non-compact type.

If a global action-angle map of the above type exists, then one can introduce the functions $f_i \in C^\infty(M)$ $(i = 1, \ldots, n)$ by the definition

$$(f_i \circ A^{-1})(Q, P) := \sum_{j=1}^{n} Q_j X(P)^{-1}_{j,i} \quad \text{with} \quad \sum_{j=1}^{n} X(P)_{i,j} X(P)^{-1}_{j,k} = \delta_{i,k}. \tag{8}$$

By using that A is a symplectomorphism, one obtains the Poisson brackets

$$\{f_i, h_j\}_M = \delta_{i,j}, \qquad \{f_i, f_j\}_M = 0. \tag{9}$$

Together with $\{h_i, h_j\}_M = 0$, these imply that the $2n$ functions $h_1, \ldots, h_n, f_1, \ldots, f_n$ are functionally independent at every point of M. The choice of any of these $2n$ functions as the Hamiltonian gives rise to a maximally superintegrable system. For example, the $(2n-1)$ independent functions $h_1, \ldots, h_n, f_2, \ldots, f_n$ (resp. $h_2, \ldots, h_n, f_1, \ldots, f_n$) Poisson commute with h_1 (resp. with f_1).

Let us now study a Hamiltonian $H \in C^\infty(M)$ of the form $H = \mathfrak{H}(h_1, \ldots, h_n)$ with some $\mathfrak{H} \in C^\infty(\mathbb{R}^n)$. Under mild conditions, this Hamiltonian is also maximally superintegrable. For example, suppose that there exists an index $l \in \{1, \ldots, n\}$ such that $\partial_l \mathfrak{H} \neq 0$ generically. For $l = 1$, the assumption guarantees that $H, h_2, \ldots, h_n$ are independent, i.e., H is Liouville integrable in the sense specified at the beginning. Next, let us choose $(n-1)$ smooth maps $V_a : \mathbb{R}^n \to \mathbb{R}^n$ $(a = 1, \ldots, n-1)$ that satisfy the equations

$$\sum_{k=1}^{n} V_a^k \partial_k \mathfrak{H} = 0 \tag{10}$$

identically on $\mathbb{R}^n$, and their values yield independent $\mathbb{R}^n$-vectors generically. In other words, the vectors $V_a(x)$ span the orthogonal complement of the gradient of $\mathfrak{H}$ at generic points $x \in \mathbb{R}^n$. Such maps V_a always exist, since the independence of their values is required only generically. By using V_a, we define the function $F_a \in C^\infty(M)$ by

$$F_a := \sum_{k=1}^{n} f_k V_a^k(h_1, \ldots, h_n), \qquad \forall a = 1, \ldots, n-1. \tag{11}$$

It is easily seen that the globally smooth functions $H, h_2, \ldots, h_n, F_1, \ldots, F_{n-1}$ are independent and they Poisson commute with H. This demonstrates that H is maximally superintegrable. Incidentally, the set $h_1, \ldots, h_n, F_1, \ldots, F_{n-1}$ also gives $(2n-1)$ independent constants of motion for H, and this holds even if we drop our technical assumption on the gradient of $\mathfrak{H} \in C^\infty(\mathbb{R}^n)$.

The above arguments are quite obvious and are well known to experts (see e.g. [13]). Among their consequences, we wish to stress the fact that *all Calogero type models possessing only scattering trajectories are maximally superintegrable.* Indeed, for these models action-angle maps of maximally non-compact type were constructed by Ruijsenaars [9]. Concretely, this is valid for the rational Calogero model, the hyperbolic Sutherland model, and for the relativistic deformations of these models due to Ruijsenaars and Schneider. The construction of the pertinent action-angle maps relies on algebraic procedures, but fully explicit formulae are not available. Therefore it might be interesting to display the maximal superintegrability of these models by alternative direct constructions of the required constants of motion.

3 Constants of motion in the rational RS model

Let us recall that the phase space of the rational Ruijsenaars-Schneider model is

$$M = \mathcal{C}_n \times \mathbb{R}^n = \{(q, p) \mid q \in \mathcal{C}_n,\, p \in \mathbb{R}^n\} \tag{12}$$

where

$$\mathcal{C}_n := \{q \in \mathbb{R}^n \mid q_1 > q_2 > \cdots > q_n\}. \tag{13}$$

The symplectic structure $\omega = \sum_{k=1}^n dp_k \wedge dq_k$ corresponds to the fundamental Poisson brackets

$$\{q_i, p_j\}_M = \delta_{i,j}, \qquad \{q_i, q_j\}_M = \{p_i, p_j\}_M = 0. \tag{14}$$

The commuting Hamiltonians of the model are generated by the (Hermitian, positive definite) Lax matrix [8] given by

$$L(q,p)_{j,k} = u_j(q,p)\left[\frac{\mathrm{i}\chi}{\mathrm{i}\chi + (q_j - q_k)}\right] u_k(q,p) \tag{15}$$

with the $\mathbb{R}_+$-valued functions

$$u_j(q,p) := e^{p_j} \prod_{m\neq j}\left[1 + \frac{\chi^2}{(q_j-q_m)^2}\right]^{\frac{1}{4}}, \tag{16}$$

where $\chi \neq 0$ is an arbitrary real coupling constant and the 'velocity of light' is set to unity.

By using the diagonal matrix $\mathbf{q} := \mathrm{diag}(q_1,\dots,q_n)$, we define the functions $I_k, I_k^1 \in C^\infty(M)$ by

$$I_k(q,p) := \mathrm{tr}\left(L(q,p)^k\right), \quad I_k^1(q,p) := \mathrm{tr}\left(\mathbf{q}L(q,p)^k\right) \qquad \forall k\in\mathbb{Z}. \tag{17}$$

It is well-known that the functions I_k pairwise Poisson commute and a convenient generating set of the spectral invariants of L is provided by the independent functions $I_1,\dots,I_n$. Note that one may also use the coefficients of the characteristic polynomial of $L(q,p)$ as an alternative generating set. The 'principal Hamiltonian' of the model is

$$h = \frac{1}{2}(I_1 + I_{-1}) = \sum_{k=1}^n \cosh(p_k)\prod_{j\neq k}\left[1+\frac{\chi^2}{(q_k-q_j)^2}\right]^{\frac{1}{2}}. \tag{18}$$

Our subsequent considerations are based on the following important formula:

$$\{I_k^1, I_j\}_M = jI_{j+k} \qquad \forall j,k\in\mathbb{Z}. \tag{19}$$

This generalizes an analogous formula found by Wojciechowski [10] in the rational Calogero model. The quantities I_j and I_k^1 form an infinite dimensional Lie algebra under the Poisson bracket, with the I_k^1 realizing the centerless Virasoro algebra:

$$\{I_k^1, I_j^1\}_M = (j-k)I_{k+j}^1 \qquad \forall j,k\in\mathbb{Z}. \tag{20}$$

We postpone the proof of the above relations for a little while.

Since $\dim(M) = 2n$ is finite, only finitely many of the functions I_k, I_m^1 can be independent. A set of $2n$ independent functions is given, for example, by

$$I_1,\dots,I_n, I_1^1,\dots,I_n^1. \tag{21}$$

To see that the Jacobian determinant

$$J := \det\frac{\partial(I_1,\dots,I_n,I_1^1,\dots,I_n^1)}{\partial(p_1,\dots,p_n,q_1,\dots,q_n)} \tag{22}$$

is non-vanishing generically, notice that J is the ratio of two polynomials in the $2n$-variables e^{p_i}, q_i $(i=1,\dots,n)$, and hence it either vanishes identically or is non-vanishing on a dense submanifold of the phase space. It is easy to confirm the non-vanishing of J in the asymptotic

region where the coordinate-differences are large. In that region $L(q,p)$ (15) becomes diagonal, which implies the leading behaviour

$$I_k \sim \sum_{i=1}^n e^{2kp_i}, \quad I_k^1 \sim \sum_{i=1}^n q_i e^{2kp_i}, \tag{23}$$

whereby the leading term of J can be calculated in terms of Vandermonde determinants. We conclude from the inverse function theorem that the $2n$ functions (21) can be serve as independent coordinates locally, around generic points of the phase space.

To elaborate the consequences of (19), let us first consider an arbitrary smooth function $I = I(I_1, \dots, I_n)$. Then we obtain from (19) that $\{\{I_k^1, I\}_M, I\}_M = 0$. This entails that I_k^1 develops linearly along the Hamiltonian flow of I,

$$I_k^1(q(t), p(t)) = I_k^1(q(0), p(0)) + t\{I_k^1, I\}_M(q(0), p(0)), \qquad \forall k \in \mathbb{Z}. \tag{24}$$

In particular, the independent functions given in (21) provide an algebraic linearization of the flow of I in the sense of [11], i.e., they can be used as explicitly given alternatives to action-angle type variables.

Next, we observe that the functions

$$I_j^1\{I_k^1, I\}_M - I_k^1\{I_j^1, I\}_M, \qquad \forall j, k \in \mathbb{Z}, \tag{25}$$

Poisson commute with I. By using these, it is possible to exhibit $(2n-1)$ smooth functions on M that Poisson commute with $I = I(I_1, \dots, I_n)$ and are functionally independent generically. For instance, for any fixed $j \in \{1, 2, \dots, n\}$ consider the functions

$$C_{k,j} := I_k^1 I_{2j} - I_j^1 I_{k+j}, \qquad k \in \{1, 2, \dots, n\} \setminus \{j\}. \tag{26}$$

By taking the $2n$ functions (21) as coordinates around generic points of M, we can easily compute the Jacobian determinant

$$J_j := \det \frac{\partial(I_a, C_{b,j})}{\partial(I_\alpha, I_\beta^1)} \quad \text{with} \quad a, \alpha \in \{1, \dots, n\},\ b, \beta \in \{1, \dots, n\} \setminus \{j\}, \tag{27}$$

and obtain that $J_j = (I_{2j})^{n-1}$, which is generically non-zero. The fact that the $(2n-1)$ functions furnished by I_m $(m = 1, \dots, n)$ and $C_{k,j}$ (26) are independent and Poisson commute with I_j shows directly that I_j is maximally superintegrable.

As another example, take the standard Ruijsenaars-Schneider Hamiltonian h in (18). The maximal superintegrability of h is ensured by the 'extra constants of motion'

$$K_j := I_j^1(I_2 - n) - I_1^1(I_{j+1} - I_{j-1}), \qquad j = 2, \dots, n. \tag{28}$$

Indeed, one has $\{K_j, h\}_M = 0$ and

$$\det \frac{\partial(I_a, K_b)}{\partial(I_\alpha, I_\beta^1)} = (I_2 - n)^{n-1}, \qquad a, \alpha \in \{1, \dots, n\},\ b, \beta \in \{2, \dots, n\}, \tag{29}$$

which guarantees that the $(2n-1)$ functions I_a, K_b are independent. Similarly, the momentum $\mathcal{P} := \frac{1}{2}(I_1 - I_{-1})$ admits the functionally independent extra constants of motion given by $L_j := I_j^1(I_2 + n) - I_1^1(I_{j+1} + I_{j-1})$ for $j = 2, \ldots, n$.

For a general $I = I(I_1, \ldots, I_n)$, let us consider functions $\mathcal{F}_a \in C^\infty(M)$ of the form

$$\mathcal{F}_a := \sum_{k=1}^{n} I_k^1 U_a^k(I_1, \ldots, I_n), \tag{30}$$

with some smooth maps $U_a : \mathbb{R}^n \to \mathbb{R}^n$ $(a = 1, \ldots, n-1)$ subject to the identity

$$\sum_{k=1}^{n} \Big(\sum_{j=1}^{n} j I_{j+k} \frac{\partial I}{\partial I_j} \Big) U_a^k = 0, \tag{31}$$

which guarantees that $\{\mathcal{F}_a, I\}_M = 0$ (cf. equations (10) and (11)). One can always choose the maps U_a $(a = 1, \ldots, n-1)$ in such a way that their values yield linearly independent $\mathbb{R}^n$-vectors at generic arguments. Then the $(2n-1)$ functions $I_1, \ldots, I_n, \mathcal{F}_1, \ldots, \mathcal{F}_{n-1}$ are independent, globally smooth and Poisson commute with I.

To summarize, we have seen that the relation (19) leads to an explicit linearization of the Hamiltonian flow associated with any $I = I(I_1, \ldots, I_n)$ and allows us to display the maximal superintegrability of the Ruijsenaars-Schneider Hamiltonian h (18) in an explicit manner (and similarly for $\mathcal{P}$ and I_j for $j = 1, \ldots, n$). The above arguments also imply, among others, the maximal superintegrability of any polynomial Hamiltonian $I = I(I_1, \ldots, I_n)$.

Now we prove the relations (19) and (20). Direct verification is possible in principle, but it would require non-trivial calculations. However, it is quite easy to obtain the claimed relations by utilizing the derivation of the rational RS model in the symplectic reduction framework presented in [14]. To explain this, we start by recalling from [14] the relevant reduction of the phase space

$$T^*GL(n, \mathbb{C}) \times \mathcal{O}(\chi) \equiv GL(n, \mathbb{C}) \times gl(n, \mathbb{C}) \times \mathcal{O}(\chi) = \{(g, J^R, \xi)\}. \tag{32}$$

Here $\mathcal{O}(\chi)$ is a minimal coadjoint orbit of the group $U(n)$, which as a set is given by

$$\mathcal{O}(\chi) := \{\mathrm{i}\chi(\mathbf{1}_n - vv^\dagger) \mid v \in \mathbb{C}^n,\ |v|^2 = n\}. \tag{33}$$

In (32) we use the trivialization of $T^*GL(n, \mathbb{C})$ by left-translations and identify the real Lie algebra $gl(n, \mathbb{C})$ with its dual space with the aid of the 'scalar product' provided by the real part of the trace

$$\langle X, Y \rangle := \Re \mathrm{tr}(XY), \qquad \forall X, Y \in gl(n, \mathbb{C}). \tag{34}$$

In terms of evaluation functions, the not identically zero fundamental Poisson brackets read

$$\{g, \langle X, J^R \rangle\} = gX, \quad \{\langle X, J^R \rangle, \langle Y, J^R \rangle\} = -\langle [X, Y], J^R \rangle, \quad \{\langle X_+, \xi \rangle, \langle Y_+, \xi \rangle\} = \langle [X_+, Y_+], \xi \rangle, \tag{35}$$

where X_+, Y_+ are the anti-hermitian parts of the constants $X, Y \in gl(n, \mathbb{C})$. The reduction is based on using the symmetry group $K := U(n) \times U(n)$, where an element $(\eta_L, \eta_R) \in K$ acts via the symplectomorphism $\Psi_{(\eta_L, \eta_R)}$ defined by

$$\Psi_{(\eta_L, \eta_R)} : (g, J^R, \xi) \mapsto (\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1}, \eta_L \xi \eta_L^{-1}). \tag{36}$$

We first set the corresponding moment map to zero, in other words introduce the first class constraints

$$J_+^R = 0 \qquad \text{and} \qquad (gJ^Rg^{-1})_+ + \xi = 0, \tag{37}$$

and then factorize by the action of K. The resulting reduced phase space can be identified with the Ruijsenaars-Schneider phase space (M, ω) (12) by means of a global gauge slice S. To describe S, we need to introduce an $\mathcal{O}(\chi)$-valued function on M by

$$\xi(q,p) := \mathrm{i}\chi(\mathbf{1}_n - v(q,p)v(q,p)^\dagger) \quad \text{with} \quad v(q,p) := L(q,p)^{-\frac{1}{2}}u(q,p), \tag{38}$$

where $L(q,p)$ is the Lax matrix (15) and $u(q,p)$ is the column vector formed by the components $u_j(q,p)$ (16). Denoting its elements as triples according to (32), the gauge slice S is given by

$$S := \{(L(q,p)^{\frac{1}{2}}, -2\mathbf{q}, \xi(q,p)) \,|\, (q,p) \in \mathcal{C}_n \times \mathbb{R}^n\}. \tag{39}$$

It has been shown in [14] that S is a global cross section of the K-orbits in the constraint-surface defined by (37), and the reduced Poisson brackets (alias the Dirac brackets) reproduce the canonical Poisson brackets (14).

Relying on the above result, we identify (M, ω) with the model S of the reduced phase space. We can then realize the functions $I_k, I_k^1 \in C^\infty(M)$ (17) as the restrictions to S of respective K-invariant functions $\mathcal{I}_k, \mathcal{I}_k^1$ on the unreduced phase space (32) furnished by

$$\mathcal{I}_k(g, J^R, \xi) := \mathrm{tr}\big((g^\dagger g)^k\big), \quad \mathcal{I}_k^1(g, J^R, \xi) := -\frac{1}{2}\Re\mathrm{tr}\big((g^\dagger g)^k J^R\big) \quad \forall k \in \mathbb{Z}. \tag{40}$$

Now the point is that the relation $\{\mathcal{I}_k^1, \mathcal{I}_j\} = j\mathcal{I}_{j+k}$ follows obviously from (35), and this implies (19) by restriction to $S \simeq M$ by using that the Poisson brackets of the K-invariant functions survive the reduction. The relation

$$\{\mathcal{I}_k^1, \mathcal{I}_j^1\} = (j-k)\mathcal{I}_{k+j}^1 \qquad \forall j,k \in \mathbb{Z} \tag{41}$$

is also easy to confirm with the aid of (35), which implies (20).

We finish with a few remarks. First, we note that the variables (21) can be useful for constructing compatible Poisson structures for the rational RS model. In this respect, see [15, 16] and references therein. Second, one may also construct RS versions of the Calogero constants of motion considered in [17]. Third, it could be interesting to study quantum mechanical analogues of the constants of motion and to characterize their algebras. It is very likely that the algebra of equations (19), (20) survives quantization. One may address this question by generalizing the method applied in [18] to quantize the analogous algebra in the Calogero case. It should be also possible to construct a quadratic algebra for the model by suitably replacing the Dunkl operators used in [18] with Dunkl-Cherednik operators. The pertinent algebras are expected to be closely related to the bispectral property of the rational RS model [19]. Finally, it is natural to ask about generalizations concerning the hyperbolic RS model and its non-relativistic limit. We plan to return to some of these issues elsewhere.

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